A NOTE ON DEFINITIONS OF INVARIANCE AND EXOGENEITY

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Ⅰ. Introduction

The renowned Lucas(1976) critique of structural econometric models has stimulated the interest in problems of invariance of variables and parameters in the context of economic policy analysis. In particular Sims(1982) reconsidered the problem of invariance of parameters of an econometric model with respect to modifications introduced to the analyzed system. His idea was further developed by Leamer(1985) in his definition of exgeneity. In a slightly different way, the concept of invariance was adopted by Engle, Hendry and Richard(1983) for defining a superexogenous variable, i.e. a variable that is unaffected, in a policy simulation analysis, by the Lucas critique.

The above concepts of invariance, being of intuitive rather of formal nature, do not produce a compact definition, from the point of view of a policy analyst, of a policy instrument variable. In section Ⅱ of this note the above concepts of invariance and their relation to the problem of exogeneity are discussed. In section Ⅲ a general mapping from the 'shallow' parameters and intervention into the parameters of interest is introduced, with the Engle, Hendry and Richard's (EHR) superexogeneity and Leamer exogeneity resulting as special cases of this mapping.

Ⅱ. Invariance and Interventions

Sims(1982) defined a parameter of an econometric model as structural, if it remains invariant under any modification of the system selected from a

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specified family of modifications. This concept was extended by Leamer(1985), in the context of selecting suitable policy instruments, for defining exogeneity of a variable x to y, if the observed conditional distribution of y given x is invariant under any modification of the system selected from a specified family of modifications that alter the process generating x.

Similar to this is the concept of superexogeneity introduced by Engle, Hendry and Richard(1983). It is grounded in three subsequent definitions of parameters invariance, structural invariance and weak exogenity. A parameter is said to be invariant for a class fo interventions, if it remains constant under these interventions. Analogously, a model is invariant for such interventions if all its parameters are invariant. Like Leamer exogeneity, structural invariance is related to a conditional model, which is said to be structural invariant if all its parameters are invariant for a class of interventions, which consist of 'any change in the distribution of the conditioning variables'. The third on the series of the EHR definitions is that of weak exogeneity; a variable x is weakly exgenous for y with respect to some parameters of interest α, if the conditional distribution of y given x contains all information for inferring about α. As a result, the EHR superexogeneity combines structural invariance and weak exogeneity; a variable x is superexogeneity for α, if it is weakly exogenous and if the conditional model of y given x, α is structurally invariant.

The concept of weak exogeneity has been fully developed in the Engle, Hendry and Richard's paper (although its practical relevance is questioned by Leamer(1985)). Nevertheless, the invariance properties of the parameters and models seem to be rather vague. Apart from semantic problem (Leamer avoid the word 'structure' and uses 'modification' in place of 'intervention'), it is still not clear what 'invariance' means in the Sims-Leamer set of definition and what the 'class of interventions'is in the EHR framework. Moreover, according to the Engle, Hendry and Richard's definition of superexogeneity, the structural invariance of a conditional model has to be decided separately from weak exogeneity.

Ⅲ. A Generalized Definition

Initially the concept of invariance can be introduced by considering the relationship between data and parameters in the 'data density' representation of an economic model. Following the Engle, Hendry and Richard's original notation, it is given by:

$$D(X^1_T \mid X_0, \Theta) = \prod_{t=1}^{T} D(x_t \mid X_{t-1}, \Theta), \qquad (1)$$

where D(·) stands for data density function, X^1_T denote the 'history' of the observed variable x_t up to time T inclusive (x_t is 1×n vector, t=1, ..., T, and X^1_T is T×n matrix), X_0 are initial conditions, X_{t-1} is information known at time (i.e. $X_{t-1}=[X_0 \ X^1_{t-1}]'$ and $\Theta \in \Theta$ is a vector of 'shallow' parameters(unknown constants)). We can partition x_t into (y_t) and 'extraneous' and potentially exgenous variable (z_t) such as:

$$x_t = [y_t, z_t],$$

and analogously

$$X^1_t = [\, Y^1_t \ z_t], X^*_t = [z_t \, X^1_{t-1}]. \qquad (2)$$

Furthermore, a class of interventions (modifications) on the variable z_t can be represented by a non-random vector δ_t , expressing the interest of a policy modeller, i.e.:

$$\overline{z_t} = z_t + \delta_t \ ,$$

(see e.g. Wallis(1984, p. 6)). It is assumed that $X^*_t \in X$, which is a subset of $R^{n \cdot t}$ and also that:

$$\overline{X_t^*} = [z_t, X_{t-1}^1] \in R^{n \cdot t}. \qquad (3)$$

Let us define many-to-one mapping from unconditional data and parameters to another set of parameters, called parameters of interest ('deep parameters') and denoted by ψ.

$$f: (\Theta, X) \rightarrow \Psi ; (\Theta, X_t^*) \rightarrow \psi = (\Theta, X_t^*). \qquad (4)$$

Let us arbitrarily select some of these mappings, having the value of $\lambda \in \Lambda$. Formalizing the EHR idea, the parameter of interest λ is said to be invariant with respect to class of interventions $\overline{X}$, if:

$$h: (\Theta, \overline{X}) \rightarrow \Lambda ; (\Theta, \overline{X_t^*}) \rightarrow \lambda = h(\Theta). \qquad (5)$$

If we partition λ into (λ_1, λ_2), $\lambda_i \in \Lambda_i$, $i=1, 2$, we can narrow the class of invariant parameters, if there exists a function ϕ such as:

$$\phi: (\Lambda_1, \overline{X}) \rightarrow \Psi ; (\lambda_1, \overline{X_t^*}) \rightarrow \psi = \phi(\lambda_1). \qquad (6)$$

The partition into λ_1, λ_2 is that for the parameters of the conditional and marginal distributions. The conditional distribution of y_t given z_t is denoted by:

$$D(y_t \mid X_t^*, X_0, \lambda_1).$$

and the marginal distribution of z_t by:

$$D(z_t \mid X_{t-1}, \lambda_2).$$

Since the only current variables in X_t^* are z_t's, they are of potential interest for a policy analyst as policy instrument variables. To deal with the Lucas critique, the parameters of the conditional model must remain unchanged for all $\overline{X_t^*} \in \overline{X}$. This leads EHR to extend the concept of invariance by introducing superexogeneity, where the variables z_t are superexogenous for ψ if, in addition to their parameters' invariance, they are weakly exogenous. Conditions for EHR weak exogeneity requires that $[(y_t \mid z_t, \lambda_1), (z_t, \lambda_2)]$ operates a classical sequential cut on $D(x_t \mid X_{t-1}, \lambda)$:

$$D(x_t \mid X_{t-1}, \lambda) = D(y_t \mid X_t^*, X_0, \lambda_1) \cdot D(z_t \mid X_{t-1}, \lambda_2),$$

$$(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2. \qquad (7)$$

If the classical sequential cut holds, invariance, weak exogeneity and superexogeneity are defined by

$$\psi = f[h^{-1}(\lambda), \overline{X_t^*}] \equiv \phi(\lambda_1). \qquad (8)$$

Relation (8) decides about the nature of the variables z_t. In particular:

(i) If (8) holds for all $\lambda \in \Lambda$, but not for all z_t, the variables z_t are weakly exogenous in the EHR sense, but λ_1 is not invariant.

(ii) If (8) holds for all z_t, but not for all λ, λ_1 is invariant but z_t is not weakly exogenous.

(iii) If (8) holds for all z_t, and for all λ, z_t is weakly exogenous and λ_1 is invariant, hence z_t is superexogenous.

In the above, (ii) and (iii) decide also about the Leamer(1985) exogeneity.

The variables z_t are exogenous in the Leamer sense, if (8) holds for all z_t, irrespective of λ.

Ⅳ. Conclusions

It seems that the above proposition is in line with the Leamer(1985) critique of the EHR concept of superexogeneity, by restricting superexogeneity (and structural invariance) to a defined class of policy interventions. In other words, a variable can be superexogenous only in a given (possible narrow) class of policy intervention and may not be superexogenous, if a policy analyst goes beyond that class. The problem of defining such a class for particular empirical can be difficult. For some simple models practical suggestion are given by Charemza and Kiraly(1988).

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국 문 요 약

약외부성, 구조적 불변 그리고 강외부성에 관해서는 Engle, Hendry, 그리고 Richard 정리는 일반화 되어져 있다. 수용가능한 정책적인 간섭와 관련이 적은 매개변수들을 관련이 깊은 매개변수로 나타내는 것을 약외부성, 구조적불변 그리고 강외부성으로서 결정되어지는 것으로 정의되어진다. 일반화된 정의는 Leamer의 견해에 있어서 정의되어지고 있고 외부성으로 이용되어지고 있다는 것을 본 논문의 분석을 통해서 설명하고 있다.