THE KUHN-TUCKER CONDITIONS AS A CHARACTERIZATION OF AN OPTIMAL SOLUTION

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Ⅰ. Introduction

Convex sets and convex functions have many special and important properties and these properties can be utilized in establishing suitable optimality conditions for nonlinear programming problems.

The purpose of this is to obtain the Kuhn-Tucker conditions under suitable convexity assumptions. This paper is divided into three sections. Section Ⅱ includes basic concepts and properties which will pave the way for the development of our arguments. And we will prove the Fritz John condition in section Ⅱ, which is important to derive our main theorems. In section Ⅲ, we will derive the main theorems. And, to illustrate how to apply our main theorems, we will give an example.

The terminologies and notations are standard and they are taken from [Bazara and Shetty, 1979]. Throughout this paper, the n-dimensional Euclidean pace is denoted by E_n

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All vectors are column vectors unless explicityly stated otherwise. Row vectors are the transpose of column vectors ; for example, x^t denotes the row vector $(x_1, x_2, \cdots, x_n)$

The norm of a vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \text{ is } (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \text{ and is denoted } \|x\| \text{ and if } x = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix},$$

then we also use the notation $x \geq 0$ to mean that $x_i \geq 0$ for all $i = 1, 2, \cdots, n$. For $S \in E_n$, the closure of S and the interior of S are denoted by cl S and int S, respectively.

Ⅱ. Preliminaries

In this section, we introduce some basic definitions and well-known results on convex sets and convex functions, which will be used in this paper. In some cases, we will omit proofs for the briefness, but we will indicate references where the propositions can be founded.

Proposition 1 : Let be a nonempty closed convex set E_n and $y \notin S$. Then there exists a nonzero vector p and a scalar α such that $p^t y > \alpha$ and $p^t x \leq \alpha$ for each $x \in S$.

Proposition 2 : Let S be a nonempty convex set in E_n and let $\bar{x} \in \partial S$, where ∂S is the boundary of S. Then there exists a nonzero vector p such that $p^t(x - \bar{x}) \leq 0$ for each $x \in cl\ S$.

Proposition 3 : Let S_1 and S_2 be nonempty convex sets in E_n and suppose $S_1 \cap S_2$ is empty. Then there exists a nonzero vector p in E_n such that

$$inf\{p^t x : x \in S_1\} > sup\{p^t x : x \in S_2\}$$

Proposition 1, 2 and 3 are proved in [Bazaraa and Shetty, 1979]

Proposition 4 : (Gordan's Theorem) : Let A be an $m \times n$ matrix. Then exactly one of the following systems has a solution. That is, if System 1 has a solution, then System 2 has no solution and if System 1 has no solution, then System 2 has a solution.

System 1. $Ax < 0$ for some $x \in E_n$

System 2. $A^t p = 0$ and $p \geq 0$ for some nonzero $p \in E_n$.

Proof : Assume that System 1 has a solution $\hat{x}$. Then, since $A\hat{x} < 0$, $\hat{p} \geq 0$ and $\hat{p} \neq 0$ we have $\hat{p}^t A \hat{x} < 0$. That is $\hat{x}^t A^t \hat{p} < 0$. But $A^t \hat{p} < 0$ by assumption. Hence $\hat{x}^t A^t \hat{p} = 0$ This is a contradiction.

Conversely, assume that System 1 has no solution. Consider the following two sets :

$S_1 = \{z : z = Ax, x \in E_n\}$

$S_2 = \{z : z < 0\}$

Note that S_1 and S_2 are nonempty convex sets such that $S_1 \cap S_2 = \emptyset$. By Proposition 3, there exists a nonzero vector p such that

$p^t Ax \geq p^t z$

for each $x \in E_n$ and $z \in S_2$

Since each component of z could be made arbitrarily large negative number, it follows that $p \geq 0$. By letting $z = 0$, we must have $\hat{p}^t A\hat{x} \geq 0$ for each $x \in E_n$ By choosing $x = -A^t p$, it follows that $-||A^t p||^2 \geq 0$ and thus $A^t p = 0$. Hence System 2 has a solution.

Definition 1 : Let S be a nonempty set in E_n and let $f : S \to E_1$ Then f is said to be differentiable at $\bar{x} \in S$ if there exist a vector $\Delta f(\bar{x})$ called the gradient vector, and a function $f : S \to E_1$ such that

$f(x) = f(\bar{x}) + \Delta f(\bar{x})^t (x - \bar{x}) + ||x - \bar{x}|| \, a(\bar{x} : x - \bar{x})$

for each $x \in S$ where $\lim_{x \to x} a(\bar{x} : x - \bar{x}) = 0$

If f is differentiable at $\bar{x}$, then there could only be one gradient vector, and this vector is given by

$$\Delta f(\bar{x}) = \left(\frac{\partial f(\bar{x})}{\partial x_1}, \frac{\partial f(\bar{x})}{\partial x_2}, \cdot \cdot \cdot, \frac{\partial f(\bar{x})}{\partial x_n} \right)^t$$

where $\frac{\partial f(\bar{x})}{\partial x_i}$, is the partial derivative of f with respect to x_i at $\bar{x}$ for $i = 1, 2, \cdots, n$

Definition 2 : Let S be a nonempty set in E_n and let $f : S \rightarrow E_1$. Then f is said to be twice differentiable at $x \in S$ if there exist a vector $\Delta f(\bar{x})$ and an n×n symmetric matrix $H(x)$, call the Hessian matrix, and a function $a : E_n \rightarrow E_1$ such that

$$f(x) = f(\bar{x}) + \Delta f(x)^t(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^t + H(\bar{x})(x - \bar{x}) + ||x - \bar{x}||^2 a(\bar{x}, x - \bar{x})$$

for each $x \in S$ where $\lim_{x \rightarrow x} a(\bar{x} : x - \bar{x}) = 0$

The entry in row i and column j of the Hessian matrix $H(\bar{x})$ is the second partial derivative $\frac{\partial^2 f(\bar{x})}{\partial x_i}$, to the main theorem.

Definition 3 : Let S be a nonempty convex set in E_n and $f : S \rightarrow E_1$

(1) The function f is said to be quasiconvex at $\bar{x} \in S$ if $f\{\lambda \bar{x} + (1 - \lambda)x \le \max\{f(\bar{x}), f(x)\}$ for each $\lambda \in (0, 1)$ and each $x \in S$ and the function f is said to be quasiconvex over S if $f\{\lambda x_1 + (1 - \lambda)x_2\} \le \max\{f(x_1), f(x_2)\}$ for each $\lambda \in (0, 1)$ and for each x_1 and $x_2 \in S$

(2) The function f is said to be pseuoconvex at $\bar{x} \in S$ if $\Delta f(\bar{x})^t(x - \bar{x}) \ge 0$ for $\bar{x} \in S$ implies that $f(x) \ge f(\bar{x})$ and, the function over f is said to be pseudoconvex over S if $\Delta f(\bar{x})^t(x - \bar{x}) \ge 0$ for each x_1 and $x_2 \in S$ implies that $f(x_2) \ge f(x_1)$

Proposition 5 : Let S is be a nonempty open convex set in E_n and let $f : S \rightarrow E_1$ be differentiable on S. If f is a convex function over S, then f is a pseuoconvex function over S.

Proof : Let $f(x_2) < f(x_1)$ By differentiability of f at x_1, for each $\lambda \in (0, 1)$ we have

$$f\{\lambda x_2 + (1 - \lambda)x_1\} - f(x_1) = \lambda \Delta f(x_1)^t(x_2 - x_1) + \lambda ||x_2 - x_1|| \, a(x_1 ; \lambda(x_2 - x_1))$$

where $a(x_1 ; \lambda(x_2 - x_1)) \rightarrow 0$ as $\lambda \rightarrow 0$. By convexity of f, we have

$$f\{\lambda x_2 + (1 - \lambda)x_1\} \le \lambda f(x_1) + (1 - \lambda)f(x_1) < f(x_1)$$

Hence the above equation implies that

$$\lambda \Delta f(x_1)^t(x_2 - x_1) + \lambda ||x_2 - x_1|| \, a(x_1 ; \lambda(x_2 - x_1)) < 0$$

Dividing by λ and letting $\lambda \rightarrow 0$ we god $\Delta f(x_1)^t(x_2 - x_1) < 0$

Proposition 6 : Let S be a nonempty open convex set in E_n and let $f : S \to E_1$ be a pseudoconvex function over S. Then S is quasiconvex function over S.

Proof : Let $f(x_2) < f(x_1)$ and suppose that $f(x_3) > f(x_1)$ for some $x_3 = \lambda x_1 + (1-\lambda)x_2$, $\lambda > 0$. Then $f(x_2) < f(x_3)$. By pseudoconvexity of f, $(x_2 - x_3)^t \Delta f(x_3) < 0$ and hence $(x_2 - x_1)^t \Delta f(x_3) < 0$.

Therefore, there exists a convex combination x_4 of x_1 and x_3 such that $(x_2 - x_1)^t \Delta f(x_4) < 0$ and $f(x_4) < f(x_3)$. It follows that $f(x_1) < f(x_4)$. By pseudoconvexity of f, we have $(x_1 - x_4)^t \Delta f(x_4) < 0$. That is, $(x_2 - x_1)^t \Delta f(x_4) > 0$. This is a contradiction.

Remark : The converse of Proposition 6 is not always true. Let $f(x) = -x^2$, $0 \le x \le 1$. Then f is a quasiconvex function. Take $x_1 = 0$ and $x_2 = \frac{1}{2}$. Then $\Delta f(x_1)^t (x_2 - x_1) \ge 0$ but $f(x_1) \ge f(x_2)$. Thus f is not a pseudoconvex function.

Now, we investigate the necessary optimality conditions for unconstrained problems.

Proposition 7 : Suppose that $f : E_n \to E_1$ is differentiable at $\bar{x}$. If there is a vector d such that $\Delta f(\bar{x}) d < 0$, then there exists $\delta > 0$ such that $f(\bar{x} + \lambda d) < f(\bar{x})$ for each $\lambda \in (0, \delta)$

Poof : By differentiability of f at $\bar{x}$ we must have

$$f(\bar{x} + \lambda d) = f(\bar{x}) + \lambda \Delta f(\bar{x})^t d + \lambda \| d \| a(\bar{x}; \lambda d)$$

where $a(\bar{x}; \lambda d) \to 0$ as $\lambda \to 0$

Rearranging the terms and dividing by λ we get

$$\frac{f(\bar{x} + \lambda d) - f(\bar{x})}{\lambda} = \Delta f(\bar{x})^t d + \| d \| a(\bar{x}; \lambda d)$$

Since $\Delta f(\bar{x})^t d < 0$ and $a(\bar{x}; \lambda d) \to 0$ as $\lambda \to 0$, there exists a $\delta > 0$ such that $\Delta f(\bar{x})^t d + \| d \| a(\bar{x}; \lambda d) < 0$ for all $\lambda \in (0, \delta)$

Corollary 1 : Suppose that $f : E_n \to E_1$ is differentiable at $\bar{x}$. If there is a vector d such that $\Delta f(\bar{x})^t d > 0$, there exists $\delta > 0$ such that $f(\bar{x} + \lambda d) > f(\bar{x})$ for each $\lambda \in (0, \delta)$

Corollary 2 : Suppose that $f : E_n \to E_1$ is differentiable at $\bar{x}$. If x is a local minimum, then $\Delta f(\bar{x}) = 0$.

Proof : Suppose that $\Delta f(\bar{x}) \neq 0$ and let $d = -\Delta f(\bar{x})$ then $\Delta f(\bar{x})^t d = \| \Delta f(\bar{x})^t \|^2 < 0$.

By Proposition 7, there is a $\delta > 0$ such that $f(\bar{x}+\lambda d) < f(\bar{x})$ for $\lambda \in (0, \delta)$. This is a contradiction to the assumption that $\bar{x}$ is a local minimum. Hence $\Delta f(\bar{x}) = 0$

Remark : The above condition uses the gradient vector whose components are the first partial of f. Hence it is a called a first-order necessary condition.

Proposition 8 : Suppose that $f: E_n \to E_1$ is differentiable at $\bar{x}$. If $\bar{x}$ is a local minimum, then $\Delta f(\bar{x}) = 0$ and $H(\bar{x})$ is positive semidefinite.

Proof : Consider an arbitrary direction d. Form differentiability of f at $\bar{x}$ we have equation (1).

(1) $f(\bar{x}+\lambda d) = f(\bar{x}) + \lambda \Delta f(\bar{x})^t d + \frac{1}{2}\lambda^2 d^t H(\bar{x}) d + \lambda^2 \|d\|^2 a(\bar{x}; \lambda d)$

where $a(\bar{x}; \lambda d) \to 0$ as $\lambda \to 0$. Since $\bar{x}$ is a local minimum, from Corollary 2, we have $\Delta f(\bar{x}) = 0$

Rearranging the terms in equation (1) and dividing by λ^2 we get

(2) $\frac{f(\bar{x}+\lambda d) - f(\bar{x})}{\lambda^2} = \frac{1}{2}\lambda^2 d^t H(\bar{x}) d + \|d\|^2 a(\bar{x}; \lambda d)$

Since $\bar{x}$ is a local minimum, $f(\bar{x}+\lambda d) \geq f(\bar{x})$ for sufficiently small λ. From equation (2), is thus clear that $\frac{1}{2}\lambda^2 d^t H(\bar{x}) d + \|d\|^2 a(\bar{x}; \lambda d) \geq 0$ for sufficiently small λ. By taking the limit as $\lambda \to 0$, it follows that $d^t H(\bar{x}) d \geq 0$

Hence $H(\bar{x})$ is positive semidefinite.

Now we give a sufficient optimality condition for unconstrained problems.

Proposition 9 : Let $f: E_n \to E_1$ be pseudoconvexity at $\bar{x}$.

If $\Delta f(\bar{x}) = 0$, then $\bar{x}$ is a grobal minimum.

Proof : Since $\Delta f(\bar{x}) = 0$, we have $\Delta f(\bar{x})^t (x - \bar{x}) = 0$ for each $x \in E_n$. By pseudoconvexity of f at $\bar{x}$, it then follows that $f(\bar{x}) \leq f(x)$ for each $x \in E_n$.

We develop a necessary optimality condition for the problem to minimize $f(x)$ subject to $g(x) \leq 0$ and $x \in S$ where $g : E_n \to E_1$

Definition 4 : Let S be a nonempty set in E_n and let $\bar{x} \in S$. The cone of feasible direction of S at $\bar{x}$, denote by D, is given by

$D = [d ; d \neq 0$ and $\bar{x} + \lambda d \in S$ for all $\lambda \in (0, \delta)$ for some $\delta > 0]$

Proposition 10 : Consider the problem to minimize $f(x)$ subject to $x \in S$ where $f: E_n \to E_1$ and S is a nonempty set in E_n. Suppose that f is differentiable at a point $\bar{x} \in S$. If $\bar{x}$ is a local optimal solution, then $F_0 \cap D = \emptyset$ where $F_0 = [d \,;\, \Delta f(\bar{x})^t d < 0]$ and D is the cone of feasible direction of S at $\bar{x}$

Proof : Suppose that there exist a vector $d \in F_0 \cap D$

By Propositon 7, there exists a $\delta_1 > 0$ such that $f(\bar{x}+\lambda d) < f(\bar{x})$ for each $\lambda \in (0, \delta_1)$. Futhermore, by Definition 4, there exists a $\delta_2 > 0$ such that $\bar{x} + \lambda d \in S$ for each $\lambda \in (0, \delta_2)$. This is a contradiction to the face that $\bar{x}$ is a local optimal solution.

In proposition 10, D is not necessarily defined in terms of the gradients of the functions involved. So, we define an open cone G_0 defined in terms of the gradients of the binding constraints at $\bar{x}$, such that $G_0 < D$.

Proposition 11 : Let $g: E_n \to E_1$ for $i = 1, 2, \cdot\cdot\cdot, m$ and $\bar{x}$ bea feasible point, and let $I = [i;\, g_i(\bar{x}) = 0]$. furthermore, suppose that f and g_i for $i \in I$ are differentiable at $\bar{x}$ and g_i for $i \notin I$ are continuous at $\bar{x}$.

If $\bar{x}$ is a local optimal solution, then $F_0 \cap G_0 = 0$, where $F_0 = [d \,;\, \Delta f(\bar{x})^t d < 0]$ and $G_0 = [d \,;\, \Delta g(\bar{x})^t d < 0]$ for each $i \in I$.

Proof : By Propositon 10, we have only to show that, $G_0 \subset D$ where D is the cone of feasible direction of the feasible region at $\bar{x}$.

Let $d \in G_0$ Since $\bar{x} \in X$ and X is open, there exists a $\delta_1 > 0$ such that

(3) $\bar{x} + \lambda d \in X$ for $\lambda \in (0, \delta_1)$

Also, since $g_i(\bar{x}) < 0$ and g_i is continuous at $\bar{x}$ for $i \notin I$, there exists a $\delta_2 > 0$ such that

(4) $g_i(\bar{x} + \lambda d) < 0$ for $\lambda \in (0, \delta_2)$ and for $i \notin I$. Since $\Delta g(\bar{x})^t d < 0$ for each $i \in I$ and by ropositon 7, there exists a $\delta_3 > 0$ such that

(5) $g_i(\bar{x} + \lambda d) < 0$ for $\lambda \in (0, \delta_3)$ and for $i \in I$

By equations (3), (4) and (5), the points of the form $\bar{x} + \lambda d$ are feasible to the problem p for each $\lambda \in (0, \delta_3)$ where $\delta = \min[\delta_1, \delta_2, \delta_3]$

Thus $d \in D$.

By using the result of Proposition 11, We derive Fritz John condition.

Proposition 12 : (Fritz John condition) : Let X be a nonempty open set in E_n and $f: E_n \to E_1$ and $g: E_n \to E_1$ for $i=1,2,\cdot\cdot\cdot,m$

Consider the Problem p to minimize $f(x)$ subject to $x \in X$ and $g_i(x) \leq 0$ for $i=1,2,\cdot\cdot\cdot,m$. Let $\bar{x}$ be a feasible solution, and let $I=\{i \ : g_i(\bar{x})=0\}$.

Furthermore, suppose that f and g_i for $i \in I$ are differentiable at $\bar{x}$ and that g_i for $i \notin I$ are continuous at $\bar{x}$. If $\bar{x}$ locally solves Problem p, then there exists scalars u_0 and u_1 for $i \in I$, such that

$$\Delta f(\bar{x}) + \sum_{i \in I} u_i \Delta g_i(\bar{x}) = 0$$

$u_0, u_i \geq 0$, for $i \in I$ and $(u_{0,} u_I) \neq (0, 0)$

where u_I is the vector whose components are u_i for $i \in I$

Furthermore, if g_i for $i \notin I$ are differentiable at $\bar{x}$, the Fritz John conditions can be written in the following equivalent form :

$$u_0 \Delta f(\bar{x}) + \sum_{i \in I}^{m} u_i \Delta g_i(\bar{x}) = 0$$

$u_i g_i(\bar{x})=0$, u_0, $u_i \geq 0$ for $i=1,2,\cdot\cdot\cdot,m$

$(u_0, u_i) \neq (0, 0)$

where u is he vector whose components are u_i for $i=1,2,\cdot\cdot\cdot,m$

Proof : Since $\bar{x}$ locally solves Problem p, by Proposition 11, there is no vector d such that $\Delta f(\bar{x})^t d < 0$ and $\Delta g(\bar{x})^t d < 0$ for each $i \in I$. Let A be the matrix whose rows are $\Delta f(\bar{x})^t$ and $\Delta g(\bar{x})^t$ for $i \in I$. The optimality condition of Proposition 11 is then equivalent to the statement that the system $A^t d < 0$ has no solution.

Hence by Proposition 4, there exists s nonzero vector $p \geq 0$ such that $A^t p = 0$. Denoting the components of p by u_0 and u_i for $i \in I$. the first part of the result follows.

The second part of the result is proved by letting $u_i = 0$ for $i \notin I$

Remark: : In the Fritz John conditions, the scalars u_0 and u_i for $i=1,2,\cdot\cdot\cdot,m$ are called Lagrangian multupliers.

The condition $u_i g_i(\bar{x})=0$ for $i=1,2,\cdot\cdot\cdot,m$ is called the complementary slackness condition.

Ⅲ. Kuhn - Tucker Conditions

With a mild additional assumption Fritz John condition reduces to Kuhn - Tucker optimality condition

Theorem 1 (Kuhn - Tucker Necessary Conditions) : Let X be a nonempty open set in E_n and let $f: E_n \to E_1$ and $g: E_n \to E_1$ for $i = 1, 2, \cdots, m$. Consider the Problem p to minimize $f(x)$ subject to $x \in X$ and $g_i \leq 0$ for $i = 1, 2, \cdots, m$. Let $\bar{x}$ be a feasible solution, and let $I = \{i : g_i(\bar{x}) = 0\}$. Suppose that f and g_i for $i \in I$ are differentiable at $\bar{x}$ and g_i for $i \notin I$ are continuous at $\bar{x}$.

Futhermore, suppose that $g_i(\bar{x})$ for $i \in I$ are linearly independent. If $\bar{x}$ locally solves problem p, then $\Delta f(\bar{x}) + \sum_{i \in I} u_i \Delta g_i(\bar{x}) = 0$.

where $u_i \geq 0$ for $i \in I$

In addition to the above assumptions, g_i for $i \notin I$ is also differntiable at $\bar{x}$, then the Kuhn - Tucker conditions could be written in the following equivalent form :

$$\Delta f(\bar{x}) + \sum_{i \in I}^{m} u_i \Delta g_i(\bar{x}) = 0$$

where $u_i g_i(\bar{x}) = 0$ for $i = 1, 2, \cdots, m$ and for $u_i \geq 0$ for $i = 1, 2, \cdots, m$

Proof : By Fritz John condition, there exist scalar u_0 and $\widehat{u}_i$ for $i \in I$, not all equal to zero, such that

$$u_0 \Delta f(\bar{x}) + \sum_{i \in I} u_i \Delta g_i(\bar{x}) = 0$$

where u_0, $\widehat{u}_i \leq 0$ for $i \in I$. Suppose that $u_0 = 0$. Then $\sum_{i \in I} u_i \Delta g_i(\bar{x}) = 0$ where $\widehat{u}_i \geq 0$ for $i \in I$ are linearly independent. By letting $u_i = \dfrac{\widehat{u}_i}{u_0}$ the first part of the theorem is proved.

The second part of the theorem is proved by letting $u_1 = 0$ for $i \in I$. Under some convex conditions, Kuhn-tucker conditions are also sufficient for optimality. This is shown below.

Theorem 2 (Kuhn-Tucker Sufficient Condition) : Let X be a nonempty open set in E_n, $f: E_n \to E_1$ and $g: E_n \to E_1$ for $i = 1, 2, \cdots, m$. Consider the Problem p to minimize $f(x)$

subject to $x \in X$ and $g_i \leq 0$ for $i = 1, 2, \cdot \cdot \cdot, m$. Let $\bar{x}$ be a feasible solution and let $I = \{i : g_i(\bar{x}) = 0\}$. Suppose that f is pseudoconvex at $\bar{x}$ and that g_i is quasiconvex and differentiable at $\bar{x}$ for each $i \in I$. Futhermore, suppose that there exist nonnegative scalars u_i for $i \in I$ such that $\Delta f(\bar{x}) + \sum_{i \in I}^{m} u_i \Delta g_i(\bar{x}) = 0$. Then $\bar{x}$ is a global optimal solution to Problem p.

Proof : Let $\bar{x}$ be a feasible solution to Problem p. Then, for each $i \in I$, $g_i(\bar{x}) \leq g_i(\bar{x})$. Since

$$g_i(\bar{x}) \leq 0 \text{ and } g_i(\bar{x}) = 0. \text{ By quasiconvexity of } g_i, \text{ we have}$$

$$\begin{aligned} g_i(\bar{x} + \lambda(x - \bar{x})) &= g_i(\lambda x + (1 - \lambda)\bar{x}) \leq \max\{g_i(x), g_i(\bar{x})\} \\ &= g_i(\bar{x}) \quad \text{for all } \lambda \in (0, 1) \end{aligned}$$

This implies that g_i does not increase by moving from $\bar{x}$ along the direction $x - \bar{x}$. By Corollary 1, we must have $\Delta g_i(\bar{x})^t (x - \bar{x}) \leq 0$. Multiplying by u_i and summing over I, we get

$$\left(\sum_{i \in I} u_i \, \Delta g_i(\bar{x})^t\right)(x - \bar{x}) \leq 0$$

But since $\Delta f(\bar{x}) + \sum_{i \in I} u_i \, \Delta g_i(\bar{x}) = 0$, it follows that $\Delta f(\bar{x})^t (x - \bar{x}) \geq 0$. Then, by pseudo-convexity of f at $\bar{x}$, we must have $f(\bar{x}) \geq f(\bar{x})$.

Now we show an example.

Consider the following problem :

$$\textit{Minimize} \quad (X_1 - \frac{9}{4})^2 + (x_2 - 2)^2$$

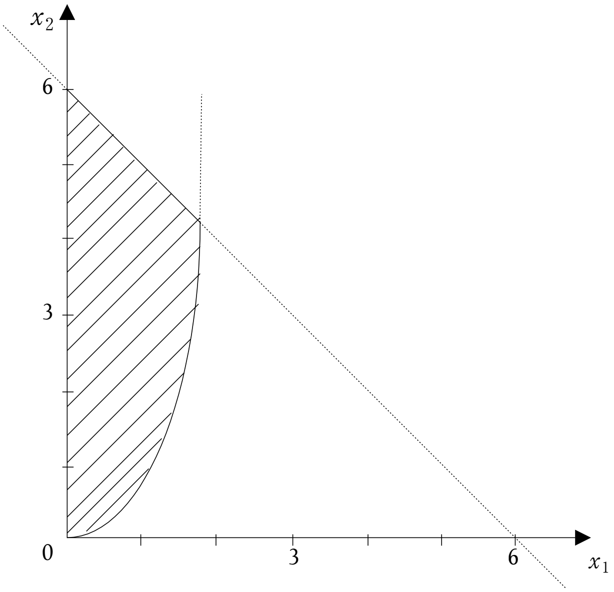
$$\begin{aligned} \textit{Subject to} \quad & x_2 - x_1^2 \geq 0 \\ & x_1 + x_2 \leq 6 \\ & x_1 \geq 0, \; x_2 \geq 0 \end{aligned}$$

The above statements are equivalent to the next statements :

$$\textit{Minimize} \quad f(x) = (X_1 - \frac{9}{4})^2 + (x_2 - 2)^2$$

$$\begin{aligned} \textit{Subject to} \quad & g_1(x) = x_2 - x_1^2 \leq 0 \\ & g_2(x) = x_1 + x_2 - 6 \leq 0 \end{aligned}$$

The feasible region is sketched in figure



We can show that Kuhn-Tucker optimality conditions are true at the point $\bar{x}=(\frac{3}{2},\frac{9}{4})^t$.

Let $I=\{i\ :\ g_i(\bar{x})=0\}=\{1\}$.

Note that f and g_i for $i = 1, 2, 3, 4$ are differentiable at $\bar{x}$ Clearly, $\nabla g_i(\bar{x})$ for $i\in I$ is linearly independent. Note that $\nabla f(\bar{x})^t=(-\frac{3}{2},\frac{1}{2})^t$ and $\nabla g_1(x)^t=(3,-1)$. Hence $\nabla f(\bar{x})+\frac{1}{2}\nabla g_1(\bar{x})=0$. Thus $u_1=\frac{1}{2}$, $u_2=0$, $u_3=0$ and $u_4=0$ satisfy the Kuhn-Tucker condition.

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<국문초록>

적절한 해결책의 특징으로서의 쿤-터커 조건

심 경 섭

본 논문의 목적은 적당한 볼록성의 가정하에서 쿤-터커 조건들을 찾아보는 것이다. 본 논문은 세 부분으로 나누어 졌는데, 논증을 발전시키는 기본 개념과 성향, 그리고 Fritz John 의 조건을 증명해 보려는 것이다. 마지막으로 주요이론을 도출하여 그 이론들이 어떻게 적용되어지고 설명되어 지는지를 예를 들어서 설명하는 것이다.