IMPLEMENTING VALID INEQUALITIES FOR UNCAPACITATED FACILITY LOCATION**

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CONTENTS

I. Introduction

The uncapacitated facility location problem (UFLP) is the problem of locating uncapacitated potential facilities so as to minimize the total cost for satisfying given demands. Among various discrete location problems, the UFLP seems to receive the most attention due to its practical and theoretical significance. The UFLP has close similarities to other classes of real world problems. Moreover, any mixed 0-1 programming methods can be applied to this problem due to its simple structure. An excellent survey on this subject can be found in Krarup and Pruzan's work.[1]

As are usual for mixed 0-1 programming problems, branch and bound algorithms are used for solving the UFLP. The computational efficiency of such branch and bound algorithms depends greatly upon how quickly it generates sharp lower bounds. Of many algorithms proposed for the UFLP, the dual-based solution method which was developed by Bilde and Krarup[2], and Erlenkotter[3] independently, has been widely accepted as the most powerful procedure. The success of this method rests on the following two points. First, this method is based on the tight UFLP formulation whose linear programming (LP) relaxation provides strong lower bounds. Second, this

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1) J. Krarup and P.M. Pruzan, "The simple plant location problem: survey and synthesis," *European Journal of Operational Research* 12 (1983) pp. 36-81.
2) O. Bilde and J. Krarup, "Sharp lower bounds and efficient algorithms for the simple plant location problem," *Ann. Discrete Math.* 1 (1977) pp. 79-97.
3) D. Erlenkotter, "A dual-based procedure for uncapacitated facility location," *Operations Research* 26 (1978) pp. 992-1009.

approach doesn't solve the LP relaxation exactly but obtains feasible solutions to its LP dual using an efficient heuristic. These dual feasible solutions, even if not optimal, still provide sharp lower bounds for the UFLP.

Although this method provides sharp lower bounds, there still exists the integrality gap which sometimes significantly increases the computational burden of the method. One possible way of reducing the integrality gap is to use valid inequalities, or cuts. Recently, a number of researches have shown that this approach is successful for solving some mixed integer programming problems. This has motivated some researchers to study on valid inequalities and facets for the UFLP. However, there hasn't been reported any attempt at incorporating valid inequalities for solving the UFLP yet.

This fact can be explained by the following reason. The usual approach of implementing valid inequalities takes the following steps: (i) solve the LP relaxation of the current problem, (ii) find a valid inequality which cuts off the LP solution, and (iii) add this inequality to the current problem and return to step (i). This approach requires the exact optimal solution of a large scale LP and that of an additional 0-1 integer problem to identify a valid inequality cutting off a given fractional solution at each step. However, such an approach might even deteriorate the whole computational efficiency, becuase of the special structure of the UFLP. Especially, the success of the dual-based heuristic makes this conjecture more credible and explains why there has been reported no attempt at implementing valid inequalities for the UFLP.[4)]

Thus, valid inequalities can't be successfully implemented, until an efficient procedure different from the usual one can be accommodated enough to overcome the difficulties described above. In this paper, we present an algorithm of incorporating valid inequalities for solving the UFLP. Heuristics of identifying the violated valid inequalities and solving the successive LP's augmented with the inequalities will be developed to minimize the computational difficulties involved. The outline of our solution procedure is as follows. We first use the dual-based procedure for obtaining the initial lower and upper bounds of the UFLP. If there exists the gap between them we generate valid inequalities to reduce the gap by using the dual feasible solution obtained through the dual-based procedure. If the above procedure fails to yield an optimal solution, we initiate a branch and bound procedure. A number of sample problems will be tested to get an information on the efficiency of our algorithm.

II. Model Formulation and Valid Inequalities

The UFLP can be formulated as the following mixed integer programming problem:

$$(P) \quad \min \sum_{i\in P}\sum_{j\in D} c_{ij}x_{ij} + \sum_{i\in P} f_i y_i, \qquad (1)$$

4) *ibid.*, pp. 992-1009.

$$\text{s.t.} \quad \sum_{i \in P} x_{ij} = 1, \quad j \in D, \tag{2}$$

$$y_i - x_{ij} \geqslant 0, \quad i \in P, j \in D, \tag{3}$$

$$x_{ij} \geqslant 0, \quad i \in P, j \in D, \tag{4}$$

$$y_i = 0 \text{ or } 1, \quad i \in P. \tag{5}$$

where y_i is 1 if facility i is established and 0 otherwise; x_{ij} is the fraction of customer j's demand supplied from facility i; f_i ($\geqslant 0$) is the fixed cost for establishing facility i, and c_{ij} is the variable cost for supplying all of customer j's demand from facility i.

Recently, several researchers have studied on valid inequalities and facets of (P) by using the fact than (P) can be transformed into the following set packing problem. In this formulation, $\overline{y_i}$ means

$$\text{(SP)} \quad \max \sum_{i \in T} \sum_{j \in D} (\theta_i - c_{ij}) x_{ij} + \sum_{i \in P} f_i \overline{y_i} - \sum_{i \in P} f_i - \sum_{j \in D} \theta_j,$$

$$\text{s.t.} \quad \sum_{i \in P} x_{ij} \leqslant 1, \quad j \in D,$$

$$\overline{y_i} + x_{ij} \leqslant 1, \quad i \in P, j \in D,$$

$$x_{ij}, \overline{y_i} = 0 \text{ or } 1, \quad i \in P, j \in D.$$

$1 - y_i$ and θ_j is a sufficiently large weight for substituting inequalities for equalities.

For exposition brevity, we use the following notation. Let G = (N, E) be the intersection graph associated with (SP). For $I^s \subseteq P$ and $J^s \subseteq D$, let $S = (s_{ij})$ be $|I^s| \times |J^s|$ 0-1 matrix with no zero column and no zero row and G^s be the subgraph of G induced by the vertices y_i, for $i \in I^s$, and x_{ij}, for $i \in I^s$ and $j \in J^s$ such that $s_{ij} = 1$. And let $\beta(G^s)$ be the covering number of G^s defined as the minimum number of plants $i \in I^s$ necessary to cover all destinations $j \in J^s$ using arcs of G^s.

Cho et al. show that the inequality

$$\sum_{i \in I^s} \sum_{j \in J^s} s_{ij} x_{ij} + \sum_{i \in I^s} (1 - y_i) \leqslant |I^s| + |J^s| - t \tag{6}$$

is a valid inequality for (P) if and only if $t \leqslant \beta(G^s)$.[5] Cho et al. also derive the necessary and sufficient condition for (6) to be a facet.[6] Cornuejols and Thizy and Guignard derived two particular families of facets of F(SP) which have the form of (6).[7]

5) D.C. Cho, E.L. Johnson, M.W. Padberg and M.R.Rao, "On the uncapacitated plant location problem I: Valid inequalities and facets," *Mathematics of Operations Research* 8 (1983) pp. 579-589.
6) D.C. Cho, M.W. Padberg and M.R.Rao, "On the uncapacitated plant location problem II: Facts and Lifting theorems," *Mathematics of Operations Research 8 (1983) pp. 590-621.*
7) G. Cornuejols and J.M. Thizy, "Facets of the location polytope," *Mathematical Programming* 23 (1982) pp. 70-74.

III. Implementation of Valid Inequalities.

In this section, in order to overcome the computational difficulties discussed before, we develop an alternative procedure for implementing valid inequalities which doesn't solve the LP relaxation exactly, but instead uses feasible solutions to its LP dual as in the dual-based procedure.[8)]

1. Cut Generation

Consider the case when inequality (6) is appended to the LP relaxation of (P) where (5) is replaced by $y_i \geqslant 0,\ i \in P$.
Then the dual of the LP relaxation of (P) with constraint (6) is given as:

$$\max \sum_{j \in D} v_j - (|J^s| - t)\gamma$$

$$\text{s.t. } v_j - w_{ij} - s_{ij}\gamma \leqslant c_{ij}, \quad i \in I^s, j \in J^s,$$

$$v_j - w_{ij} \leqslant c_{ij}, \quad i \in P - I^s \text{ or } j \in D - J^s,$$

$$\sum_{j \in D} w_{ij} + \gamma \leqslant f_i, \quad i \in I^s,$$

$$\sum_{j \in D} w_{ij} \leqslant f_i, \quad i \in P - I^s,$$

$$w_{ij}, \gamma \geqslant 0, \quad i \in P, j \in D,$$

where v_j, w_{ij} and γ are the dual variables corresponding to (2), (3) and (6) respectively.

For any feasible choice of v_j, and γ, each variable w_{ij} may be set at its lowest possible value as

$$w_{ij} = \begin{cases} \max(0, v_j - c_{ij} - s_{ij}\gamma), & \text{if } i \in I^s, j \in J^s, \\ \max(0, v_j - c_{ij}), & \text{otherwise,} \end{cases}$$

which will maintain the feasibility while keeping the objective value as high as possible. Then the dual problem may be replaced by the following condensed form:

$$\text{(DA)} \quad \max \sum_{j \in D} v_j - (|J^s| - t)\gamma \tag{7}$$

$$\text{s.t. } \sum_{j \in D} \max(0, v_j - c_{ij} - s_{ij}\gamma) + \sum_{j \in D - J^s} \max(0, v_j - c_{ij}) + \gamma \leqslant f_i, \ i \in I^s, \tag{8}$$

$$\sum_{j \in D} \max(0, v_j - c_{ij}) \leqslant f_i, \quad i \in P - I^s, \tag{9}$$

$$\gamma \geqslant 0. \tag{10}$$

Remark 1. If $\gamma = 0$, (DA) is equivalent to the so-called condensed dual, denoted by (D).

8) Erlenkotter, *op. cit.*, pp. 992-1009.

Using the special structure of the dual, they develop an efficient heuristic which provides near optimal solutions to it. Now we consider a way of still using its special structure when implementing valid inequalities to solve (P). Let $z_d(\gamma)$ be the optimal objective value of (DA) for any fixed $\gamma \geqslant 0$.

Remark 2. Suppose that any optimal solution of the LP relaxation of (P) has fractional value, and that we have a valid inequality (6) cutting off that optimal fractional solution. Then if (DA) assumes the LP dual of (P) with the cut, it is easily shown that there exists some $\gamma > 0$ such that $z_d(\gamma) \geqslant z_d(0)$. And the inequality strictly holds if the LP fractional solution is the unique optimal one.

Now we consider how to select an inequality (6) which provides a better dual objective value when being added to the LP relaxation. This is to determine a matrix S whose corresponding inequality (6) provides the optimal objective value of (DA) greater than $z_d(0)$.

Proposition 1. There exists some $\gamma > 0$ such that $z_d(\gamma) > z_d(0)$, if and only if

$$z'_d(0^+) = \lim_{\Delta\gamma \to 0^+} \frac{z_d(\Delta\gamma) - z_d(0)}{\Delta\gamma} > 0. \tag{11}$$

Proof. It is obvious from the fact that $z_d(\gamma)$ is a piecewise linear concave function.

(11) holds if $z_d(\Delta\gamma) > z_d(0)$ for $\Delta\gamma > 0$ and sufficiently small. Thus our strategy is to select a cut whose corresponding (DA) provides $z_d(\Delta\gamma) > z_d(0)$ for sufficiently small $\Delta\gamma$.

Throughout the process, exactly speaking, the approximate of $z_d(\gamma)$ is considered, since only the dual feasible solutions, instead of the optimal one, are generated. However, even if approximate is used, the whole process gives ever increasing lower bounds of (P). In addition, when calculating the approximate of $z_d(\gamma)$, we impose the following restriction on the search of v-vector: v_j's are to be kept non-decreasing as γ increases. This restriction is for the computational efficiency of the whole solution procedure, because it simplifies not only the cut identification but also the successive computation of the augmented LP's. Under this restriction, it is easily resolved by solving a specially constructed problem to check whether (11) holds or not for a given cut.

For any feasible solution $\{v_j\}$ of (D), let $J^s(i) = \{j \in J^s : v_j > c_{ij} \text{ and } s_{ij} = 1\}$ for $i \in I^s$, and define

$$SL_i = f_i - \sum_{j \in D} \max(0, v_j - c_{ij}), \ i \in P,$$

$$a_{ij} = \begin{cases} 1, \text{if } v_j \geqslant c_{ij}, \text{ for } i \in P, j \in D, \\ 0, \text{otherwise,} \end{cases}$$

$$b_i = \begin{cases} L, & \text{if } SL_i > 0, \text{ for } i \in P, \\ |J^s(i)| - 1, & \text{if } SL_i = 0 \text{ for } i \in I^s, \\ 0, & \text{if } SL_i = 0, \text{ for } i \in P - I^s, \end{cases}$$

where L is a sufficiently large weight not less than $|D|$. Consider the following problem:

$$z_s(v) = \max \sum_{j \in D} \sigma_j$$

$$\text{s.t.} \sum_{j \in J^s} a_{ij} \max(0, \sigma_j - k_{ij}) + \sum_{j \in D\text{-}J^s} a_{ij}\sigma_j \leqslant b_i, \; i \in P, \tag{12}$$

$$\sigma_j \geqslant 0, j \in D,$$

where for $i \in I^s$ and $j \in J^s$,

$$k_{ij} = \begin{cases} 1, & \text{if } v_j = c_{ij} \text{ and } s_{ij} = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1. Suppose that we have an optimal solution $\{v_j^+\}$ of (D), i.e., (DA) with $\gamma = 0$ and an inequality (6). Let $\{\sigma_j^+\}$ be an optimal solution of the problem (12) corresponding to that inequality and $\{v_j^+\}$. If

$$z_s(v^+) > (|J^s| - t), \tag{13}$$

then $z_d'(0^+) > 0$.

Proof. Note that $|J^s| - t \geqslant 0$ since $t \leqslant \beta(G^s) \leqslant |J^s|$. By the definition of a_{ij} and b_i there exists some $\gamma^+ > 0$ such that $\{v_j^+ + \sigma_j^+\gamma^+\}$ is feasible to (DA). Since not all σ_j^+'s are zero,

$$z_d(\gamma^+) - z_d(0) = \sum_{j \in D} \sigma_j^+\gamma^+ - (|J^s| - t)\gamma^+$$

$$= \{z_S(v^+) - (|J^s| - t)\}\gamma^+ > 0.$$

Since (12) has a highly specialized structure, an efficient heuristic for finding a feasible solution satisfying (13) can be developed and will be shown later.

2. The Branch and Bound Algorithm

Based on the observations discussed in the preceding section, we develop two procedures, one for identifying a valid inequality and the other for successively solving an LP augmented by the identified inequality. The two procedures also use the dual feasible solutions of the LP relaxation as in the dual-based method.

We first solve (D), the condensed dual of (P) using Erlenkotter's dual-based heuristic and obtain a feasible solution of (D) and an integer feasible solution of (P). If there exists a gap between them, we initiate the cut implementation procedure. Cuts are selected and implemented only when (DA) corresponding to a selected cut provides a dual feasible solution having better objective than that of the current feasible solution of (D), and thus the implementation of valid inequalities guarantees the monotone improvement of the dual objective value. Furthermore, cuts are successively so generated that the J^S's corresponding to the obtained cuts never overlap. This makes it possible to easily obtain the feasible solutions for the LP and of (P) with more than one cut. Our procedure is heuristic and thus more elaborated method can be devised for generating and implementing cuts. However, as discussed, a simple approach seems to be adequate considering that the additional effort might sharply increase the computing time of the whole process.

The first part of our algorithm is a procedure for selecting an inequality (6) whose resulting (DA) gives positive z'_d (0+). The ideal way of identifying a cut is to directly select an S such that the corresponding value of z_s (v) is greater than $|J^S| - t$. However, if possible, this might require the computation of a large scale 0-1 integer problem even more complex than (P). Thus our cut identification method, based on this observation, consists of the following three steps: (i) select a candidate matrix S, (ii) calculate t, the lower bound of $\beta(G^S)$, and (iii) solve (12) corresponding to the inequality (6) associated with S to check whether (13) holds.

We develop a procedure for selecting a possible S using the special structure of (12). We first select $J^S \subseteq D$, and then determine $I^S \subseteq P$ and s_{ij} for $i \in I^S$ and $j \in J^S$. Consider the following observations.

Remark 3. Given an optimal solution $\{v_j^+\}$ of (D) and an inequality (6), the following properties hold.

(i) Let $I^* = \{i \in P : SL_i = 0\}$, then the constraints of (12) for $i \in P - I^*$ are redundant.
(ii) If $v_j^+ > c_{ij}$ for some $i \in I^* - I^S$, any feasible solution of (12) satisfies $\sigma_j = 0$.
(iii) If $|J^S(i)| \leqslant 1$ for some $i \in I^S$, any feasible solution of (12) satisfies $\sigma_j = 0$, for all $j \in D$ with $v_j \leqslant c_{ij}$.

Based on (i) and (ii) and the definition of $J^S(i)$, $I^S \subseteq P$ and s_{ij} for $i \in I^S$ and $j \in J^S$ are constructed by the following strategy to maintain b_i's as large as possible.

$$I^S = \{i \in I^* : v_j > c_{ij} \text{ for some } j \in J^S\}, \tag{14}$$

$$s_{ij} = \begin{cases} 1, & \text{if } v_j > c_{ij}, \\ 0, & \text{otherwise.} \end{cases} \tag{15}$$

Then the remaining problem is how to select J^S, because it directly affects the value of z'_d (0+) through z_s (v), $|J^S|$ and t. Although the relation among them is difficult to clearly itemize, we develop the following scheme of constructing J^S to reflect this relation as well as possible:

(C1) $|J^S(i)| \geqslant 2$ for all $i \in I^S$,
(C2) $|I^S(j)| \geqslant 2$ for all $j \in J^S$ where $I^S(j) = \{i \in I^S : v_j > c_{ij} \text{ and } s_{ij} = 1\}$.

(C1) is for considering Remark 3-(iii). And (C2) is based on the conjecture that any $j \in D - J^S$ such that $v_j > c_{ij}$ for only a single $i \in I^S$ has little chance to have a positive effect on z'_d (0+) when being added to J^S. This conjecture is due to the following reason: if J^S is augmented by some $j \in D - J^S$ such that $v_j > c_{ij}$ for only a single $i \in I^S$, $|J^S|$ increases by one and $z_s(v)$ can increase at most by one since b_i increases by one, and the covering number is unlikely to increase especially under (C1). It is interesting that (C1) and (C2) are also the conditions which Cho et al. has shown to be satisfied by a facetial inequality of (P).

After S is determined, we need to caluculate the $\beta(G^S)$ or its lower bound. $\beta(G^S)$ can be obtained by solving the following problem.

$$\beta(G^S) = \max\{\sum_{i\in I^S} z_i : \sum_{j\in I^S} s_{ij}z_i \geqslant 1, j \in J^S, z_i = 0 \text{ or } 1, i \in I^S\}. \tag{16}$$

This problem is a set covering problem and several solution approaches have been proposed for it. However, we calculate a lower bound of the optimal objective value of (16) by using a dual feasible solution of its LP relaxation. This seems to be adequate when considering the computational efficiency of the whole solution procedure. Moreover, the lower bounds of $\beta(Gs)$ can also be used for constructing an inequality (6) and these lower bounds are made to be tight using the integral condition of $\beta(G^S)$.

The LP relaxation of (16) where the constraints z_i = 0 or 1 is replaced by $z_i \geqslant 0$ can be dualized as follows:

$$\max\{\sum_{j\in J^S} p_j : \sum_{j\in J^S} s_{ij}p_j \leqslant 1, i \in I^S, p_j \geqslant 0, j \in J^S\}. \tag{17}$$

We generate a feasible set $\{p_j\}$ through a simple heuristic and obtain a lower bound of $\beta(G^S)$ by letting $t = \lceil \Sigma_{j\in J^S}\, p_j \rceil$ where $\lceil a \rceil$ means the smallest integer not less than a. The proposed heuristic is as follows:

(i) INITIALIZE : $p_j = \min\limits_{\substack{i\in I^S \\ s_{ij}=1}} \dfrac{1}{\Sigma_{j\in J^S} s_{ij}}$, for all $j \in J^S$.

(ii) For each $j \in J^S$: UPDATE $p_j \leftarrow p_j + \min_{\substack{i\in I^S \\ s_{ij}=1}} (1 - \Sigma_{j\in J^S}\, p_j)$.

An inequality (6) can be made more successful by the lifting procedure. One possible lifting is to increase the value of some s_{ij} with zero to one as far as the resulting inequality is valid. This can be done using a feasible solution $\{p_j\}$. For a given feasible choice of p_j, we reset s_{ij}'s with zero value to one as many as possible without violating the feasibility of (17). Considering the structure of (12), the priority for selecting s_{ij}'s for resetting is given to the ones with $v_j \geqslant c_{ij}$. We choose s_{ij}'s with $v_j = c_{ij}$ as the candidates for lifting, since s_{ij}'s with $v_j > c_{ij}$ are already set to be 1.

Now we develop an algorithm to solve problem (12). By Remark 3-(i) and the strategy of determining S, problem (12) can be reformulated as the following problem with smaller size.

$$\max \sum_{j\in D} \sigma_j$$

$$\text{s.t.} \sum_{j\in J^S} a_{ij} \max\{0, \sigma_j - k_{ij}\} + \sum_{j\in D-J^S} a_{ij}\sigma_j \leqslant |J^S(i)| - 1, i \in I^S, \tag{18}$$

$$\sum_{j \in D - J^S} a_{ij}\sigma_j \leqslant 0, \quad i \in I^* - I^S, \quad \sigma_j \geqslant 0, \quad j \in D.$$

Problem (18) is still large, since even relatively simple real problems have a large number of customers. We avoid computational difficulties by allowing only a limited number of σ_j's to have a positive value. Let $\hat{J}^S = \{j \in D - J^S : v_j \geqslant c_{ij}$ for only a single $i \in I^S$ and $v_j < c_{ij}$ for any other $i\}$. The variables to be allowed to increase are σ_j's corresponding to $j \in J^S \cup \hat{J}^S$.

The reasoning for this selection can be partly explained by the following observation. Between the LP relaxation of (P) and its condensed dual the following complementary slackness relationships exist.

$$y_i \, [f_i - \sum_{j \in D} \max\{0, v_j - c_{ij}\}] = 0, \quad i \in P \tag{19}$$

$$(y_i - x_{ij}) \max(0, v_j - c_{ij}) = 0, \qquad i \in P, j \in D, \tag{20}$$

The duality gap between the optimal integer solution and the optimal dual solution of the relaxed LP is usually due to the violation of (20). Consider the case when an inequality (6) is added to (P). Then (20) changes to the following condition.

$$(y_i - x_{ij}) \max(0, v_j - c_{ij} - s_{ij}\gamma) = 0, j \in I^S, j \in J^S,$$

$$(y_i - x_{ij}) \max(0, v_j - c_{ij}) = 0, i \in P - I^S \text{ or } j \in D - J^S,$$

Since σ_j determines the increase of v_j, a desirable solution $\{\sigma_j\}$ is such that its resulting solution of (DA) reduces the duality gap. The increase of v_j for $j \in \hat{J}^S$ doesn't deteriorate the complementary slackness violations. Moreover, under our policy for selecting S, the increase of v_j for $j \in J^S$ is offset by that of γ. $|J^S|$ is usually small compared with $|D|$, and thus once σ_j's for $j \in D - J^S$ are fixed, (18) reduces to a problem with conformable size.

Then the following heuristic is developed to solve (18):

1. INITIALIZE: $\sigma_j = 0$ for $j \in D$ and $b_i = |J^S(i)| - 1$ for $i \in I^S$.
2. ASSIGN $\sigma_j = \min(1, b_i/\Sigma_{j \in J^S} s_{ij}), j \in \hat{J}^S$; UPDATE $b_i \leftarrow b_i - \Sigma_{j \in D}\sigma_j, i \in I^S$.
3. ASSIGN $\sigma_j = \min_{i \in I^S} (b_i/\Sigma_{j \in J^S} s_{ij}), j \in J^S$; UPDATE $b_i \leftarrow b_i - \Sigma_{j \in D}\sigma_j, i \in I^S$.
4. FOR each $j \in J^S$: UPDATE $\sigma_j \leftarrow \sigma_j + \min_{\substack{i \in I^S \\ s_{ij}=1}} b_i$ and $b_i \leftarrow b_i - s_{ij}\sigma_j, i \in I^S$.

Note that the whole process explained so far is for the case that only a single inequality is added to (P). If (P) is augmented by more than one inequality, the LP dual of the problem is slightly different from (DA). Considering that the main idea developed in this section is still valid for a case where more than one inequality exists and that cuts are implemented successively one by one, the slightly modified version of our algorithm can be used for this case. However, this not only involves many computational difficulties in carrying each step but also requires excessive memory. Therefore we generate cuts that the J^S's corresponding to the obtained cuts don't overlap. Thus even in a case with more than one cut, the whole process proposed here is directly used except that D is replaced by $D - \bigcup_k J^{s^k}$ where S^k is the matrix corresponding the k^{th} identified cut.

If we find a feasible solution $\{\sigma_j\}$ which gives the objective value greater than $|J^s| - t$, we augment this inequality with (P) and solve (DA) to obtain a better lower bound. Otherwise we proceed to select another inequality. As already described, if an inequality (6) satisfies the first case, there exists some $\gamma > 0$ such that $\{v_j + \sigma_j\gamma, \gamma\}$ is feasible to (DA) corresponding to this inequality and it gives a better objective value than that of the current solution v_j of (D). The only needed thing is to calculate a maximal value of γ as far as it satisfies the feasible condition of (DA).

Suppose we have a feasible solution $\{v_j^+\}$ of (D), and $\{\sigma_j^+\}$ of (18), then it is sufficient to find a maximal value of γ satisfying the following condition:

$$SL_i^s(\gamma) = f_i - \sum_{j \in J^s} \max(0, v_j^+ + \sigma_j^+\gamma - c_{ij} - s_{ij}\gamma)$$

$$- \sum_{j \in \hat{J}^s} \max(0, v_j^+ + \sigma_j^+\gamma - c_{ij}) - \gamma \geqslant 0, \; i \in I^s,$$

$$SL_i^s(\gamma) = f_i - \sum_{j \in J^s} \max(0, v_j^+ + \sigma_j^+\gamma - c_{ij})$$

$$- \sum \max(0, v_j^+ + \sigma_j^+\gamma - c_{ij}) - \gamma \geqslant 0, \; i \notin I^s.$$

Using the fact that $SL_i^s(\gamma)$ is a piecewise linear concave function and that $SL_i^s(0) \geqslant 0$, we computer $\gamma^* = \max\{\gamma | SL_i^s(\gamma) \geqslant 0, \; i \in I^s\}$.

Now we briefly describe how the procedure of implementing valid inequalities proposed here is incorporated into a branch and bound solution method for solving (P). We first solve the LP relaxation of (P) by directly using Erlenkotter's dual-based method which consists of the dual ascent and adjustment procedure. These two procedures provide a feasible solution of (D). Erlenkotter also proposes a primal procedure to construct an integer feasible solution using an obtained dual feasible solution. However we adopt a slightly different primal procedure since his method makes a computational difficulty when cuts are added to (P). We first select I* as a candidate open facility set as in Erlenkotter's method, but use a 'drop' heuristic to determine a final open facility set.

These dual and primal procedures provide lower and upper bounds of (P). If there exists a gap between them, we select a valid inequality (6) using a current dual feasible solution. As described, cuts are generated that corresponding J^s's never overlap. This procedure is repeated until the procedure finds an optimal solution or until no more columns remain for constructing a cut. If the whole procedure fails to yield an optimal solution, we proceed to a branch and bound phase.

IV. Computational Results

The solution procedure was coded in FORTRAN IV and run on CYBER 170-845. We have conducted computational experiments on a wide variety of problems some of which are appeared in the literature. However, since the first part of our solution method is the same as Erlenkotter's method, here are shown the problems for which his method fails to provide an optimal solution without branching.

The problems with dimensions of (5 x 8) and (33 x 33) are the ones used in Erlenkotter[9)], and the problems with dimensions of (8 x 30), (8 x 40) and (12 x 30) are constructed by extracting the first period data from the Roodman-Schwarz's dynamic problems.[10)] The remaining problems with dimensions of (10 x 40) and (30 x 60) are randomly generated.

First, relatively smaller problems were tested to compare the lower bounds obtained by implementing cuts generated here with those without cuts, the optimal integer objective value, and the LP optimal value. Since our algorithm generates cuts so that the J^S's corresponding to the generated cuts are not overlapped, the number of generated cuts depends on the size of the problem solved. However, Table 1 shows that even a small number of cuts effectively provide improved lower bounds.

Table 1. Comparison of Lower Bounds.

Problem size	Lower bound			LP opt.	Integer opt.
	without cut	with cut	Number of cuts		
5 x 8	1540	1565	1	1547.5	1565
5 x 8	1560	1570	1	1565	1580
5 x 8	1590	1597	1	1597	1615
5 x 8	1630	1725	1	1677.5	1725
8 x 30	19359	19380	1	19359	19533
8 x 30	19409	19409	1	19409	19633
8 x 30	19823	19835	1	19834	20133
8 x 40	24077	24117	2	24101	24243
8 x 40	24692	24705	2	24699	24923
12 x 30	7939	7950	3	7943	8180
12 x 30	8076	8088	2	8080	8330
12 x 30	8157	8171	3	8171	8380

9) *ibid.*, pp. 992-1009.
10) G.M. Roodman and L.B. Schwarz, "Optimal and heuristic facility phase out strategies," AIIE Transactions 7 (1975) pp. 177-184.

As already mentioned, there exists the trade-off relation between the quality of an obtained lower bound and the computational load needed to get it. To show the efficiency of our algorithm in terms of the whole computation, several problems including some large ones are tested and CPU times are compared with those for DUALOC (Erlenkotter's dual-based algorithm) known to date as the ultimatum. See Table 2.

Table 2. Computational Results.

Problem size	Ours				DUALOC		
	lower bound	Number of cuts	Number of nodes	CPU (secs.)	lower bound	Number of nodes	CPU (secs.)
8 x 30	19380	2	5	0.107	19359	5	0.162
8 x 30	19409	1	5	0.122	19409	7	0.157
8 x 30	19479	3	5	0.134	19471	11	0.202
8 x 30	19835	2	9	0.241	19823	7	0.191
8 x 30	21365	3	7	0.284	20962	11	0.425
8 x 40	24705	5	5	0.208	24692	9	0.418
8 x 40	24117	3	7	0.162	24077	7	0.131
8 x 40	24495	1	3	0.107	24485	7	0.371
8 x 40	25068	2	9	0.223	25056	11	0.410
8 x 40	25144	10	15	0.616	25209	7	0.582
12 x 30	7514	7	9	0.367	7514	13	0.465
12 x 30	7950	6	5	0.249	7939	15	0.613
12 x 30	8088	6	7	0.297	8076	13	0.454
12 x 30	8171	9	7	0.391	8157	9	0.377
12 x 30	8452	7	11	0.434	8429	13	0.549
10 x 40	7702	11	7	0.627	7695	9	0.632
10 x 40	7922	3	5	0.276	7905	5	0.309
33 x 33	20313	2	5	0.208	20340	5	0.225
33 x 33	21163	1	3	0.086	21099	5	0.140
30 x 60	8688	13	11	1.240	8677	13	1.498
30 x 60	8754	11	11	1.195	8748	11	1.461
30 x 60	8936	9	9	0.914	8809	13	1.597
30 x 60	8866	9	9	0.874	8857	15	1.764

V. Conclusions

In this paper, we have presented an algorithm of incorporating valid inequalities for solving the UFLP. Although there have been reported the successful results of implementing valid inequalities for solving some combinatorial problems, this approach seems to be hardly applicable to the UFLP. This is mainly due to the fact that the special structure of the UFLP makes the traditional cut implementation method inefficient since that method can't take advantage of the inherent structure of the UFLP.

As an effort to overcome the obstacle described above we have developed several heuristics of identifying the violated valid inequalities and solving the successive linear programming relaxation augmented with the inequalities in order to fully exploit the structural properties of the UFLP. Those heuristics are constructed to use the dual feasible solutions of the LP relaxation as in the dual-based procedure which is known to be the best one for solving the UFLP.

Although the proposed algorithm has been proved to be efficient through computing experiments with a number of sample problems, the additional improvement seems to be still possible. The improvement may be achieved by constructing some devices to more easily identify valid inequalities in good quality.

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<국문요약>

유효부등식을 이용한 공급능력에 제한이 없는 설비의 입지선정 문제에 관한 연구

명 영 수

교통 및 통신시스템 등과 같은 대규모 시스템에서의 설비입지 선정문제는 그 현실적 요구에 의해 경영과학 분야의 가장 중요한 문제 중 하나로서 인식되어 왔다. 이러한 설비입지 선정문제 중에서도 특히 공급능력에 제한이 없는 설비입지 선정문제 (Uncapacitated Facility Location Problem)는 현실문제에의 광범위한 응용 가능성과 0－1 혼합형정수계획 문제로서의 이론적 중요성 때문에 그 동안 많은 연구가 이루어져 왔다.

본 연구의 목적은 공급능력에 제한이 없는 설비입지 선정문제의 응용 가능성을 더욱 넓히기 위하여 이 문제에 대한 효율적인 해법을 개발하는것이다. 이를 위하여 기존에 이론적 연구에만 그쳤던 대상문제의 유효부등식을 실제 해법에 적용할 수 있는 방법을 제시하였다. 여타 0-1 정수계획 문제에서의 성공사례에도 불구하고 본 연구의 대상문제에는 유효부등식이 적용되지 못하였는데, 본 연구에서는 원문제의 쌍대 구조를 이용한 휴리스틱을 개발함으로써 이를 가능케 하였다. 특히 이러한 접근방법이 성공적이었음이 개발된 해법의 효율성 test 를 통해 입증되었다.